Annexure VII
UNIVERSITY GRANTS COMMISSION
BAHADUR SHAH ZAFAR MARG
NEW DELHI 110 002
PROFORMA FOR SUBMISSION OF INFORMATION AT
THE TIME OF SENDING THE
FINAL REPORT OF THE WORK DONE ON THE PROJECT

1. Title of the Project: Modified de Broglie-Bohm Approach to Quantum Cosmology

2. NAME AND ADDRESS OF THE PRINCIPAL INVESTIGATOR: Moncy V. John, Vilavinal, Omalloor 689647, Kerala

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4. UGC APPROVAL LETTER NO. AND DATE: MRP(S) 1207/11-12/KLMG022/UGC-SWRO

5. DATE OF IMPLEMENTATION ....

6. TENURE OF THE PROJECT ....

7. TOTAL GRANT ALLOCATED ....

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9. FINAL EXPENDITURE ....

10. TITLE OF THE PROJECT: Modified de Broglie-Bohm Approach to Quantum Cosmology

11. OBJECTIVES OF THE PROJECT ....

Our attempt in the proposed project was to apply the methods of the modified de Broglie-Bohm approach to quantum mechanics in to a variety of cosmological models and to identify models having the expected classical correspondence. More specifically, the objectives were

(a) To formulate the wave equation for the universe (the Wheeler-DeWitt equation) in a form suitable for the modified de Broglie-Bohm approach to quantum mechanics.

(b) To solve this equation for various cosmological models, defined in terms of their energy content, equation of state, etc.

(c) To find the classical limit of these quantum cosmological formulations in order to identify models with correct classical limit.

12. WHETHER OBJECTIVES WERE ACHIEVED:

In this work we found a potential that provides a solution to the WD equation such that the evolution of the universe can be considered classical. For this, we resorted to trajectory representations and searched for a potential that gives identical classical and quantum evolution for $a(t)$. An equivalent condition for exact quantum-classical correspondence is the vanishing of the quantum potential. We have found that in a Friedmann model where the total energy density varies as $1/a^n$, the exact correspondence happens for $n = 2$. In this case, the WD equation has a simple solution of the form $\Psi = \exp(ima^2/2\hbar)$ and both the dBB and MdBB representations give the result that $a \propto t$. We see that for this kind of total energy/matter density, the classical evolution is also $a \propto t$. Therefore this solution is unique, having identical classical and quantum evolution, as in the case of free particles described by plane waves in ordinary quantum mechanics.

13. ACHIEVEMENTS FROM THE PROJECT:

The wave equation for the universe (the Wheeler-DeWitt equation) was formulated in a form suitable for the modified de Broglie-Bohm approach to quantum mechanics. This equation was solved for various cosmological models, defined in terms of their energy content, equation of state, etc. The classical limit of these quantum cosmological formulations were found in order to identify models with correct classical limit.

14. SUMMARY OF THE FINDINGS:

Writing down the WD equation $\hat{H}\Psi = 0$ in a mini-superspace is one of the simplest applications of quantum theory in problems involving gravity. The dBB and MdBB schemes show that in ordinary quantum mechanics of a single particle, a plane wave solution of the Schrodinger equation corresponds to the motion of a particle that can be considered classical, having a trajectory with constant momentum. This plane wave solution, obtained for a free particle with $V = 0$, is used to describe incoming particles in quantum scattering problems. In this paper we attempt to find a potential that provides a solution to the WD equation such that the evolution of the universe can be considered classical.

For this, we resort to trajectory representations and search for a potential that gives identical classical and quantum evolution for $a(t)$. An equivalent condition for exact quantum-classical correspondence is the vanishing of the quantum potential. We have found that in a Friedmann model where the total energy density varies as $1/a^n$, the exact correspondence happens for $n = 2$. In this case, the WD equation has a simple solution of the form

$\Psi = \exp(ima^2/2\hbar)$ and both the dBB and MdBB representations give the result that $a \propto t$. We see that for this kind of total energy/matter density, the classical evolution is also $a \propto t$. Therefore this solution is unique, having identical classical and quantum evolution, as in the case of free particles described by plane waves in ordinary quantum mechanics. The evolution is called 'coasting' in the literature and is usually associated with the classic 'Milne model', which describes an empty universe. However, the present model is not empty; it is a Friedmann model with $\rho \propto 1/a^2$, capable of describing the expansion of the present universe to a very good approximation.

The approach in this work is very similar to that in Gao and Cai (2014), who got the evolution of the scale factor in a quantum cosmological model, computed using the dBB formalism. Their results are significant, for it presents a rigorous proof that the universe could be spontaneously and irreversibly created from nothing, and that after-wards it proceeds to an inflationary epoch. It is also shown that the exponential expansion stops when the scale factor grows sufficiently large, no matter whether it is closed, open, or flat. It is of particular interest to note that for the open case in this scenario, after the inflationary epoch, the universe enters a coasting evolution $a \propto t$ and continues to expand in this fashion. The spontaneous creation of the coasting universe obtained in the present paper from nothing is an attractive problem worth pursuing. This special case would have benefits such as no singularity, creation of particles by the decay of vacuum energy, a low dark matter/dark energy budget for the present universe, etc.

Solutions having exact classical-quantum correspondence, as obtained in this work, are important for they save us from the conceptual difficulties in applying the quantum theory to the entire universe.

15. CONTRIBUTION TO THE SOCIETY:

Just as quantum mechanics contains classical mechanics as a limiting case, a quantum gravity theory must have the general theory of relativity as its classical limit. In line with these, it is natural to require that the classical limit of quantum cosmology (which is the quantum theory applicable to the universe) must be a robust cosmological model capable of describing the state of the universe during its present epoch. In quantum cosmology, the universe is taken to be described by a wave function, which obeys the Wheeler-DeWitt (WD) equation. However, there are some deeper conceptual problems in this theory when compared to standard quantum mechanics. One such problem is related to the use of probability in this scheme. The WD equation does not contain the time parameter and is written as $\hat{H}\Psi = 0$. In the much

simplified version of a mini-superspace model in quantum cosmology, this equation is similar to a zero energy Schrodinger equation and leads to a stationary wave function independent of time. Since there are no excited states in this case, the superposition of energy eigenfunctions cannot be done. This makes the application of the superposition principle, and consequently that of the probability axiom, meaningless. Another problem is that the role of the observer in this scheme is not clear. It suffers a basic difficulty that the observer of this quantum system is part of the system itself, which is the universe, and is not just an outside 'classical' observer. Consequently there cannot be any 'wave function collapse' as in standard quantum mechanics. It is hoped that a proper resolution of these problems can give impetus to a deeper understanding of cosmology.

16. WHETHER ANY PH.D. ENROLLED/PRODUCED OUT OF THE PROJECT: NO

17. NO. OF PUBLICATIONS OUT OF THE PROJECT: ONE (M. V. John, *Exact Classical Correspondence in Quantum Cosmology*, Gravitation and Cosmology 2015, Vol. 21, No. 3, pp. 208-215. Pleiades Publishing, Ltd., 2015.)

Paper attached.

(PRINCIPAL INVESTIGATOR)   (PRINCIPAL)
                                              (Seal)

Exact Classical Correspondence in Quantum Cosmology

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Abstract—We find a Friedmann model with appropriate matter/energy density such that the solution of the Wheeler-DeWitt equation exactly corresponds to the classical evolution. The well-known problems in quantum cosmology disappear in the resulting coasting evolution. The exact quantum-classical correspondence is demonstrated with the help of the de Broglie-Bohm and modified de Broglie-Bohm approaches to quantum mechanics. It is reassuring that such a solution leads to a robust model of the universe, which agrees well with cosmological expansion indicated by SNe Ia data.

1. INTRODUCTION

Just as quantum mechanics contains classical mechanics as a limiting case, a quantum gravity theory must have general relativity as its classical limit. In line with this, it is natural to require that the classical limit of quantum cosmology (which is a quantum theory applicable to the universe) must be a robust cosmological model capable of describing the state of the universe during its present epoch. In quantum cosmology, the universe is taken to be described by a wave function which obeys the Wheeler-DeWitt (WDW) equation [1]. However, there are some deeper conceptual problems in this theory when compared to standard quantum mechanics. One such problem is related to the use of probability in this scheme. The WDW equation does not contain the time parameter and is written as $\hat{H}\Psi = 0$. In the much simplified version of a mini-superspace model in quantum cosmology, this equation is similar to a zero energy Schrödinger equation and leads to a stationary wave function independent of time. Since there are no excited states in this case, the superposition of energy eigenfunctions cannot be done. This makes the application of the superposition principle, and consequently that of the probability axiom, meaningless. Another problem is that the role of the observer in this scheme is not clear. It suffers a basic difficulty that the observer of this quantum system is part of the system itself, which is the universe, and is not just an outside "classical" observer. Consequently, there cannot be any "wave function collapse" as in standard quantum mechanics. It is hoped that a proper resolution of these problems can give an impetus to a deeper understanding of cosmology.

We restrict ourselves to the mini-superspace model in quantum cosmology. The work proposes to identify a Friedmann model with an appropriate energy/matter content in the WDW equation, such that it has exact classical correspondence. The idea of exact classical correspondence is introduced in the sense that the WDW equation reduces to the classical Hamilton-Jacobi-type equation in cosmology under the substitution $\Psi = \exp i\hat{S}/\hbar$. We find that the energy/matter density in the universe with a particular equation of state parameter makes the WDW equation equivalent to the classical equation. Obviously, since in this scenario the universe is always classical, there is no need to invoke the probability axiom. Similarly, there is no need of wave function collapse, etc., for the universe. Most important is the fact that the resulting cosmological model has very good predictions for the present universe.

The equivalence of classical and quantum evolution is more rigorously shown by making use of the de Broglie-Bohm (dBB) [2–5] and the modified de Broglie-Bohm (MdBB) [6] approaches to quantum mechanics. Recently, there has been a renewed interest in quantum trajectories in real space, such as the dBB trajectories and those due to Floyd, Faraggi and Matone (FFM) [7–9]. These real trajectory representations could provide a sensible, ontological interpretation to several quantum phenomena. MdBB was the result of a similar attempt, where complex quantum trajectories were first conceived and drawn by modifying the dBB approach. Such trajectories were drawn in [6] for the cases of a harmonic oscillator, a particle approaching a potential step, a free

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particle, a time-dependent spreading wave packet etc. It was found that MdBB has several advantages over the conventional dBB trajectories, where there are problems such as stationarity of particles in bound eigenstates etc. [4]. Our attempt in this paper would be to apply the methods of both dBB and MdBB quantum trajectories to quantum cosmology and to study the quantum-classical correspondence for a class of models, which procedure helps us to identify a cosmological model having the desired exact correspondence.

We note that in quantum cosmology obtaining the evolution of the scale factor $a(t)$ using the dBB approach is already familiar. A recent work [10], which uses this method, shows that the universe can spontaneously and irreversibly be created from nothing, and also that it can then pass over to an inflationary epoch. The authors first obtain solutions to the WDW equation for an empty universe, with closed, open or flat space sections, when a particular operator ordering is chosen in it. The evolutions of the scale factor $a(t)$ in the early universe, corresponding to these solutions, are found using the dBB formalism. The present work follows a similar approach. The prospects of relating one of these early evolutions to the late classical "coasting" evolution obtained in the present work seems to be an interesting open problem, and we discuss it in the last section.

The paper is organized in the following way. In Section 2 we review the basics of the dBB and the MdBB trajectory representations in quantum mechanics. In Section 3, the theoretical framework of the Friedmann models in classical cosmology is outlined. The Wheeler-De Witt (WDW) equation in quantum cosmology is introduced in the next section. In Section 5, using the dBB and MdBB approaches, we analyze the behavior of a class of Friedmann models during the quantum to classical transition. The results are discussed in the last section.

2. QUANTUM TRAJECTORIES

2.1. Classical Mechanics and the Analogy with Wave Optics

A well-known formulation of classical mechanics of a system of particles, other than the Lagrangian and Hamiltonian formulations, is based on solving the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) = 0, \qquad (1)$$

where $S(q_i, \alpha_i, t)$ is Hamilton's principal function, H is the Hamiltonian, q_i are the configuration space variables, α_i are integration constants, and t denotes time. If the Hamiltonian is independent of t and is a

constant of motion equal to the total energy E, then S and Hamilton's characteristic function W are related by

$$S = W - Et. \qquad (2)$$

Since W is time-independent, the surfaces $W = $ const have fixed locations in configuration space. However, the $S = $ const surfaces move in time and may be considered as wave fronts propagating in this space [11]. For a single particle one can show that the wave velocity at any point is given by

$$u = \frac{E}{|\nabla W|} = \frac{E}{p} = \frac{E}{mv}. \qquad (3)$$

This shows that the velocity of a point on this surface is inversely proportional to the spatial velocity of the particle. Also one can show that the trajectories of the particle must always be normal to the surfaces of constant S. The particle momentum is obtained as [11]

$$\mathbf{p} = \nabla W. \qquad (4)$$

Thus the particle trajectories orthogonal to surfaces of constant S in classical mechanics, as we have discussed above, are similar to light rays traveling orthogonal to Huygens' wavefronts. It was this analogy with wave optics that de Broglie utilized to formulate his principle of wave-particle duality.

2.2. De Broglie-Bohm Quantum Trajectory Formulation

The fact that de Broglie's 1924 PhD thesis [12] which contained a seed of a new mechanics which can replace classical mechanics, did not get due attention for a long time. In it, and later in a paper published in 1927 [2], he proposed that Newton's first law of motion be abandoned and replaced by a new postulate, according to which a freely moving body follows a trajectory which is orthogonal to the surfaces of equal phase of an associated guiding wave. He presented his new theory in the 1927 Solvay conference too, but it was not well received at that time [13].

The Schrödinger equation, which is the cornerstone of quantum mechanics, can be recast in a form reminiscent of the Hamilton-Jacobi equation by substituting $\Psi = R \exp(iS/\hbar)$ in it. This results in two equations. The first one is

$$\frac{\partial S}{\partial t} + \left[\frac{1}{2m}(\nabla S)^2 + V\right] = \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}, \qquad (5)$$

which resembles the classical Hamilton-Jacobi equation. The second one is

$$\frac{\partial \rho}{\partial t} + \nabla\left(\rho\frac{\nabla S}{m}\right) = 0, \qquad (6)$$

which is the standard continuity equation obeyed by Born's probability density $\rho \equiv \Psi^\star \Psi = R^2$. The guidance relation for particle trajectories, as proposed by de Broglie, is

$$m_i \dot{\mathbf{x}}_i = \nabla_i S. \qquad (7)$$

Solving this, we get a particle trajectory for each initial position. In 1952, David Bohm rediscovered this formalism [3] in a slightly different way. He noted that if we take the time derivative of the above equation, Newton's law of motion may be obtained in the modified form

$$m\ddot{\mathbf{x}}_i = -\nabla_i(V + Q). \qquad (8)$$

Here,

$$Q = -\Sigma \frac{\hbar^2}{2m_i} \frac{\nabla_i^2 |\Psi|}{|\Psi|} \qquad (9)$$

is called the "quantum potential". This results in the same mechanics as that of de Broglie, but in the language of Newton's equation with the above unnatural quantum potential. We shall note that the first-order equation of motion (7) suggested by de Broglie represented a unification of the principles of mechanics and optics, and this should be distinguished from Bohm's second-order dynamics based on (8) and (9) [13]. Bohm's revival of de Broglie's theory in the pseudo-Newtonian form has led to a mistaken notion that the de Broglie-Bohm (dBB) theory constituted a return to classical mechanics. In fact, de Broglie's theory was a new formulation of dynamics in terms of wave-particle duality.

It is well known that neither the particle picture nor the trajectories have any role in the standard Copenhagen interpretation of quantum mechanics and that several physicists, including Albert Einstein, were not happy with the Copenhagen interpretation for various reasons. Einstein was not happy with the de Broglian mechanics either [13]. The reason for the latter was speculated to be the problem it faced with real wave functions with the phase gradient $\nabla S = 0$. For a wide class of bound state problems, the time-independent part of the wave function is real, and hence while applying the equation of motion (7), the velocity of the particle turns out to be zero everywhere. This feature, that the particles in bound eigenstates are at rest irrespective of their position and energy, is counter-intuitive and is not satisfactory.

2.3. Complex Quantum Trajectories

The modified dBB approach also attempts to unify the principles of mechanics and optics, but in a different way. An immediate application of the formalism was to solve the above problem of stationarity of quantum particles in bound states in a natural way [6].

For obtaining this representation, we shall substitute $\Psi = e^{i\hat{S}/\hbar}$ in the Schrödinger equation to obtain a single equation, which is now called the quantum Hamilton-Jacobi equation (QHJE) [11]

$$\frac{\partial \hat{S}}{\partial t} + \left[\frac{1}{2m} \left(\frac{\partial \hat{S}}{\partial x} \right)^2 + V \right] = \frac{i\hbar}{2m} \frac{\partial^2 \hat{S}}{\partial x^2}. \qquad (10)$$

In this section, we restrict ourselves to one dimension. An equation of motion for the particle, similar to that used by de Broglie, can now be defined:

$$m\dot{x} \equiv \frac{\partial \hat{S}}{\partial x} = \frac{\hbar}{i} \frac{1}{\Psi} \frac{\partial \Psi}{\partial x}. \qquad (11)$$

The trajectories $x(t)$ are obtained by integrating this equation with respect to time, with various initial positions. In general, they lie in a complex x-plane where $x \equiv x_r + ix_i$. We find that even real wave functions produce nonstationary trajectories. The reason for this is that the identification $\Psi = e^{i\hat{S}/\hbar}$ helps one to utilize all information contained in Ψ for obtaining the trajectory. (In contrast, the dBB approach, which uses $\Psi = Re^{iS/\hbar}$, does not have this advantage.) The observable trajectory of the particle may be considered to be the real part $x_r(t)$ of the complex trajectories [6].

When $\hat{S} = \hat{W} - Et$, where E and t are assumed to be real, the Schrödinger equation gives us an expression for the particle energy

$$E = \frac{1}{2} m\dot{x}^2 + V(x) + \frac{\hbar}{2i} \frac{\partial \dot{x}}{\partial x}. \qquad (12)$$

The last term resembles the quantum potential in the dBB theory. However, the concept of quantum potential is not an integral part of this formalism since the equation of motion (11) adopted here is not based on it.

The complex eigen trajectories of a free particle, harmonic oscillator and potential step problems and complex trajectories for a wave packet solution were obtained in [6]. As an example, complex trajectories in the first excited state of a harmonic oscillator are shown in the figure.

This formalism was extended to three-dimensional problems such as the hydrogen atom by Yang [14] and was used to investigate one-dimensional scattering problems and bound state problems by Chou and Wyatt [15]. Later, a complex trajectory approach for solving the QHJE was developed by Tannor and co-workers [16]. Sanz and Miret-Artes found the complex trajectory representation useful in better understanding the nonlocality in quantum mechanics [17]. The role of probability in this scheme was explained and an extension of the probability density to the complex plane was performed in [18]. Certain
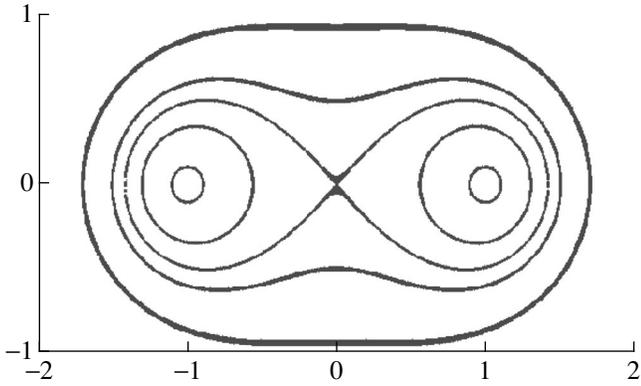
Fig. 1. Complex trajectories for the first excited state of a harmonic oscillator.

characteristics, such as identical classical and MdBB quantum trajectories for particles in coherent states, were pointed out in [19].

3. CLASSICAL COSMOLOGY

3.1. Field Equations as Euler-Lagrange Equations

The Einstein equations in cosmology can be obtained as Euler-Lagrange equations using an action principle. For the maximally spatially symmetric spacetime described by the Robertson-Walker (RW) metric [20]

$$ds^2 = c^2 dt^2$$
$$- a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \right] \quad (13)$$

under the ADM 3+1 split [21], the action is

$$I = 2\pi^2 c^2 \int N a^3 \left[-\frac{1}{16\pi G} \left(\frac{6}{N^2} \frac{\dot{a}^2}{a^2} - \frac{6kc^2}{a^2} \right) \right.$$
$$\left. + \frac{\dot{\phi}^2}{2N^2} - V(\phi) \right] dt \equiv \int L \, dt. \quad (14)$$

This is when the universe contains only a spatially homogeneous scalar field ϕ. Here N is called the lapse function, for which one can fix some convenient gauge. Using the Lagrangian L, we may write the Euler-Lagrange equations for the variables N and ϕ and obtain the Einstein equations in this case as

$$\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} = \frac{8\pi G}{3c^2} \left[\frac{\dot{\phi}^2}{2} + V(\phi) \right], \quad (15)$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{dV(\phi)}{d\phi} = 0, \quad (16)$$

by fixing the gauge $N = 1$.

Similarly, for a de Sitter model which contains only a cosmological constant, the Lagrangian can be taken to be

$$L = 2\pi^2 N a^3 c^2$$
$$\times \left[-\frac{1}{16\pi G} \left(\frac{6}{N^2} \frac{\dot{a}^2}{a^2} - \frac{6kc^2}{a^2} \right) - \rho_\lambda \right]. \quad (17)$$

The Einstein equations

$$\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho_\lambda, \quad (18)$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} = 8\pi G\rho_\lambda. \quad (19)$$

are obtained by writing the Euler-Lagrange equations corresponding to variations with respect to N and a, in the gauge $N = 1$.

3.2. Hamiltonian Formulation

As in the above, we consider a model in which the only degrees of freedom are those of the scale factor a of a RW spacetime and a spatially homogeneous scalar field ϕ. The Lagrangian for this problem is given by (14) as

$$L = -\frac{3\pi}{4G} N c^2 \left[\frac{a\dot{a}^2}{N^2} - kc^2 a \right.$$
$$\left. - \frac{8\pi G}{3} \left(\frac{\dot{\phi}^2}{2N^2} - V(\phi) \right) a^3 \right]. \quad (20)$$

From this we can find the conjugate momenta π_a and π_ϕ to a and ϕ as

$$\pi_a = \frac{\partial L}{\partial \dot{a}} = -\frac{3\pi}{2G} c^2 \frac{a\dot{a}}{N} \quad (21)$$

and

$$\pi_\phi = \frac{\partial L}{\partial \dot{\phi}} = 2\pi^2 c^2 \frac{a^3 \dot{\phi}}{N}. \quad (22)$$

The canonical Hamiltonian can now be constructed as

$$\mathcal{H}_c = \pi_a \dot{a} + \pi_\phi \dot{\phi} - L = N \left[-\frac{G}{3c^2} \frac{\pi_a^2}{a} \right.$$
$$\left. - \frac{3\pi}{4G} kc^4 a + \frac{3\pi}{4G} a^3 \left(\frac{G}{3\pi^3 c^2} \frac{\pi_\phi^2}{a^6} + \frac{8\pi G}{3} V(\phi) \right) \right]$$
$$\equiv N\mathcal{H}. \quad (23)$$

The secondary constraint now gives

$$\mathcal{H} = -\frac{G}{3\pi c^2} \frac{\pi_a^2}{a} - \frac{3\pi}{4G} kc^4 a$$
$$+ \frac{3\pi}{4G} a^3 \left(\frac{G}{3\pi^3 c^2} \frac{\pi_\phi^2}{a^6} + \frac{8\pi G}{3} V(\phi) \right) = 0, \quad (24)$$

which is equivalent to (15). For the RW spacetime which contains only a cosmological constant, (17) helps us to write the constraint equation as

$$\mathcal{H} = -\frac{G}{3\pi c^2}\frac{\pi_a^2}{a} - \frac{3\pi}{4G}kc^4 a$$
$$+ \frac{3\pi}{4G}a^3\frac{8\pi G}{3}\rho_\lambda = 0. \qquad (25)$$

This equation is equivalent to (18). The fact that $\mathcal{H} = 0$ is a consequence of a new symmetry of the theory, namely, time reparametrization invariance. This means that using a new time variable t' such that $dt' = N\,dt$ will not affect the equations of motion. This also enables one to choose some convenient gauge for N.

4. WHEELER-DEWITT EQUATION IN QUANTUM COSMOLOGY

Canonical quantization of a classical system like the one above means introduction of a wave function $\Psi(a, \phi)$ and requiring that it satisfies [21]

$$i\hbar\frac{\partial\Psi}{\partial t} = \mathcal{H}_c\Psi = N\mathcal{H}\Psi. \qquad (26)$$

To ensure that time reparametrization invariance is not lost at the quantum level, the conventional practice is to ask that the wave function is annihilated by the operator version of $\mathcal{H}$, i.e.,

$$\mathcal{H}\Psi = 0. \qquad (27)$$

Eq. (27) is called the Wheeler-DeWitt (WDW) equation.

This equation is analogous to a zero energy Schrödinger equation, in which the dynamical variables a, ϕ etc. and their conjugate momenta π_a, π_ϕ etc. are replaced by the corresponding operators. The wave function Ψ is defined on the minisuperspace with just one coordinate a, and we expect it to provide information regarding the evolution of the universe. An intriguing fact here is that the wave function is independent of time; they are stationary solutions in minisuperspace.

The transition from the quantum cosmology era, which is the earliest epoch after the Big Bang, to the late classical era is expected to happen when the scale factor a exceeds the Planck length $l_p = \sqrt{\hbar G/c^3}$, which is a quantity with dimension of length constructed using the three fundamental constants. The usual approach in quantum cosmology to study this transition is to check whether the wave function of the universe Ψ is strongly peaked around the trajectories $a(t)$ identified by the classical solutions [22]. The wave functions commonly arising in quantum cosmology are of WKB form and may be broadly classified as oscillatory, of the form $e^{iS/\hbar}$ or exponential, of the form $e^{-I/\hbar}$. The wave function of the form $e^{iS/\hbar}$ is normally thought of as being peaked about a set of solutions to the classical equations and hence predicts a classical behavior. A wave function of the form $e^{-I/\hbar}$ predicts no correlation between coordinates and momenta and so cannot correspond to a classical behavior. If Born's probability axiom is assumed to be valid in the case of the universe, one can argue that Ψ must be strongly peaked around the trajectories identified by the classical solutions.

5. CLASSICAL CORRESPONDENCE IN QUANTUM COSMOLOGY

The application of Born's probability is criticized on the grounds that the role of the observer and the observed is not clear in this case. As mentioned in the Introduction, the observer is part of the observed system, which is the universe. This is unlike the situation in standard quantum mechanics where the quantum system is always observed by an outside classical observer. Moreover, since we have only the zero-energy ground state solution for the WDW equation (27), the superposition principle cannot work, and therefore an application of the probability axiom is meaningless.

Instead, in this paper we approach the problem of classical correspondence by rewriting the WDW equation as the quantum Hamilton-Jacobi equation in cosmology and then analyzing the quantum trajectories obtained from it. The quantum cosmological Hamilton-Jacobi equation (QCHJE) (which is akin to QHJE obtained from the Schrödinger equation) can be derived by substituting $\Psi = e^{i\hat{S}/\hbar}$ in the WDW equation. Instead, if we can identify $\hat{S}$ from the solution of the WDW equation, quantum trajectories can be found by integrating a de Broglie-type equation of motion. If the classical and quantum trajectories are found to be identical, there is an exact classical correspondence. We do not expect this to happen for all potentials. The above is an acceptable criterion for exact classical correspondence since what we try to find is whether the WDW equation in this case tends to the corresponding classical equation and is devoid of $\hbar$.

It is instructive to take the free particle ($V = 0$) case in ordinary quantum theory as an example. Substituting $\psi = e^{i\hat{S}/\hbar}$ into the Schrödinger equation, one gets the QHJE as (See Sec. 2.3)

$$\left[\frac{1}{2m}(\nabla\hat{S})^2 + V\right] + \frac{\partial\hat{S}}{\partial t} = \frac{i\hbar}{2m}\nabla^2\hat{S}. \qquad (28)$$

This would be the classical Hamilton-Jacobi (HJ) equation for Hamilton's principal function $\hat{S}$ if the

right-hand side (the quantum potential) vanishes. A more rigorous approach to obtain exact quantum-classical correspondence would be to see whether the quantum trajectories obtained in a particular problem are the same as its classical trajectories. One can see that in the above case in ordinary quantum mechanics, plane waves in the dBB/MdBB schemes result in rectilinear motion, which is free particle motion in classical mechanics. In other words, the plane-wave solution can be used to describe free particles as if they were classical particles, and this justifies the use of plane waves to describe incident particles in quantum scattering problems. The dBB scheme predicts identical classical and quantum trajectories only in this zero potential region. MdBB shows such property for the ground-state harmonic oscillator and for several coherent states [19, 23].

In quantum cosmology, for finding a potential that results in exact classical correspondence, we note that the Lagrangian for the Friedmann model of the universe is of the form

$$L = 2\pi^2 a^3 N c^2 \left[-\frac{1}{16\pi G} \left(\frac{6}{N^2} \frac{\dot{a}^2}{a^2} - \frac{6kc^2}{a^2} \right) - \frac{C_n}{a^n} \right]. \quad (29)$$

In the Friedmann model, the total density of the universe varies as $\rho \propto \frac{1}{a^n}$. The index n is related to the equation of state parameter w (that appears in the assumed relation $p = w\rho c^2$) as $n = 3(1 + w)$. A pressureless, dust-filled universe has $w = 0$ and hence $n = 3$. For a universe filled with relativistic matter, $w = 1/3$ and $n = 4$. For a universe which contains only vacuum energy with $w = -1$, we have $n = 0$ and hence the energy density remains a constant.

Varying L with respect to N, the lapse function, and a, the scale factor, we get, respectively,

$$\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} = \frac{8\pi G}{3} \frac{C_n}{a^n}, \quad (30)$$

and

$$\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} = \frac{8\pi G}{3} \frac{C_n(3-n)}{a^n}, \quad (31)$$

which are the relevant classical constraint equations in this case. The conjugate momentum to a may now be defined as

$$\Pi_a = \frac{\partial L}{\partial \dot{a}} = -\frac{3\pi}{2G} c^2 a \dot{a}. \quad (32)$$

The Hamiltonian is

$$H = \Pi_a \dot{a} - L$$
$$= -\frac{G}{3\pi c^2} \frac{\Pi_a^2}{a} - \frac{3\pi}{4G} c^4 ka + 2\pi^2 c^2 \frac{C_n}{a^{n-3}}. \quad (33)$$

The WDW equation can be written by making the operator replacements for Π_a and a in H. However, finding the operator corresponding to Π_a^2/a is problematic due to an operator ordering ambiguity. The most commonly used form is

$$\frac{\Pi_a^2}{a} \rightarrow -\hbar^2 a^{-r-1} \frac{\partial}{\partial a} \left(a^r \frac{\partial}{\partial a} \right), \quad (34)$$

where the choice of r is arbitrary and is usually made according to convenience. Using this expression with $r = 0$, we obtain the WDW equation as

$$\frac{d^2\Psi}{da^2} - \left(\frac{9\pi^2 kc^6}{4G^2\hbar^2} a^2 - \frac{6\pi^3 c^4}{G\hbar^2} \frac{C_n}{a^{n-4}} \right) \Psi = 0. \quad (35)$$

The corresponding quantum cosmological Hamilton-Jacobi equation (QCHJE) is obtained by putting $\Psi = \exp(i\hat{S}/\hbar)$ in the above. This gives

$$\left(\frac{d\hat{S}}{da} \right)^2 + \left(\frac{9\pi^2 kc^6}{4G^2} a^2 - \frac{6\pi^3 c^4}{G} \frac{C_n}{a^{n-4}} \right)$$
$$= i\hbar \frac{d^2\hat{S}}{da^2}. \quad (36)$$

Another frequent choice used in (34) is $r = -1$, which results in

$$\frac{d^2\Psi}{da^2} - \frac{1}{a} \frac{d\Psi}{da}$$
$$- \left(\frac{9\pi^2 kc^6}{4G^2\hbar^2} a^2 - \frac{6\pi^3 c^4}{G\hbar^2} \frac{C_n}{a^{n-4}} \right) \Psi = 0. \quad (37)$$

This leads to a QCHJE of the form

$$\left(\frac{d\hat{S}}{da} \right)^2 + \left(\frac{9\pi^2 kc^6}{4G^2} a^2 - \frac{6\pi^3 c^4}{G} \frac{C_n}{a^{n-4}} \right)$$
$$= i\hbar \left(\frac{d^2\hat{S}}{da^2} - \frac{1}{a} \frac{d\hat{S}}{da} \right). \quad (38)$$

When equations (36) and (38) are compared with the classical Hamilton-Jacobi equation in this case, it is seen that the second terms on their left-hand sides correspond to the classical potential and the terms on the right-hand sides correspond to the quantum potential $Q(a)$. Without the latter terms, these equations reduce to the classical Hamilton-Jacobi equation for the Friedmann model. What we look for is a classical potential that can give identical quantum and classical trajectories. It is easily seen that the vanishing of the quantum potential is also a necessary condition for exact classical-quantum correspondence.

We note that both the quantum cosmological Hamilton-Jacobi equations (36) and (38) are nonlinear differential equations, with no apparent simple solutions. However, the latter form of QCHJE turns

out to have an exact solution for the case with $n = 2$, such that Ψ is given by

$$\Psi(a) \propto \exp\left(\pm i \frac{ma^2}{2\hbar}\right). \qquad (39)$$

Here we have defined a constant m as

$$m = \frac{3\pi}{2G} c^2 \left(\frac{8\pi G}{3} C_2 - kc^2\right)^{1/2}. \qquad (40)$$

Note that, as in the case of plane waves in quantum mechanics for a single free particle, the wave function of the form (39) allows the dBB (where $\Psi \equiv R \exp(iS/\hbar)$, with R, S real) and MdBB (where $\Psi \equiv \exp(i\hat{S}/\hbar)$) schemes to give identical solutions for $a(t)$ too. That is, one can identify the same function $ma^2/2$ as S and $\hat{S}$, respectively, in these schemes. This leads to the same de Broglie-type equation of motion in them, which can be obtained from

$$\Pi_a = -\frac{3\pi}{2G} c^2 a\dot{a} = \frac{\partial \hat{S}}{\partial a} = \pm ma. \qquad (41)$$

Here we have used Eq. (32). Integrating this gives the time evolution of the scale factor a. The solution is

$$a = \mp \frac{2G}{3\pi c^2} mt = \mp \left(\frac{8\pi G}{3} C_2 - kc^2\right)^{1/2} t, \quad (42)$$

which is the well-known "coasting evolution" in cosmology. This result, obtained from the dBB/MdBB quantum trajectory approaches, agrees with the classical evolution. One can see this by putting $n = 2$ in the classical constraint equation (30) and integrating for the trajectories. Thus we have exact quantum-classical correspondence for this Friedmann model with the total energy density varying as $1/a^2$.

We also check the behavior of the quantum potential $Q(a)$ in this case. It may be noted that for the operator ordering $r = -1$,

$$Q(a) = -i\hbar \left(\frac{d^2\hat{S}}{da^2} - \frac{1}{a}\frac{d\hat{S}}{da}\right) \qquad (43)$$

is equal to zero, given the above solution of $\hat{S} = ma^2/2$. This further shows the desired feature of identical classical and quantum evolution for $n = 2$.

The solution of the form (42) commonly appears in the Milne model [24], but since this is an empty model devoid of any matter/energy, much attention was not drawn towards it. The present model has the evolution of the scale factor coinciding with that of the Milne model, but it is not an empty one. This is a Friedmann model with the total energy density varying as $1/a^2$. It may be argued that also such an energy, with the equation of state parameter $w =$ $-1/3$, is unrealistic. But it was noted in [25] that this model can describe the universe when we have two components for the energy density of the universe; one of which is ordinary matter (with $w = 0$ or $1/3$), and the other is a decaying vacuum energy (with $w = -1$). It has been shown that such a model has very close predictions agreeing with recent apparent magnitude-redshift observations of Type Ia supernovae and is devoid of the conceptual problems that were noted in standard cosmology during the 1980s [25–27].

6. DISCUSSION

Writing down the WDW equation $\hat{H}\Psi = 0$ in a minisuperspace is one of the simplest applications of quantum theory in problems involving gravity. The dBB and MdBB schemes show that in ordinary quantum mechanics of a single particle, a plane-wave solution of the Schrödinger equation corresponds to the motion of a particle that can be considered classical, having a trajectory with constant momentum. This plane-wave solution, obtained for a free particle with $V = 0$, is used to describe incoming particles in quantum scattering problems. In this paper we attempt to find a potential that provides a solution to the WDW equation such that the evolution of the universe can be considered classical.

For that, we resort to trajectory representations and search for a potential that gives identical classical and quantum evolution for $a(t)$. An equivalent condition for an exact quantum-classical correspondence is the vanishing of the quantum potential. We have found that in a Friedmann model where the total energy density varies as $1/a^n$, the exact correspondence happens for $n = 2$. In this case, the WDW equation has a simple solution of the form $\Psi = \exp(ima^2/2\hbar)$, and both the dBB and MdBB representations give the result that $a \propto t$. We see that for this kind of total energy/matter density, the classical evolution is also $a \propto t$. Therefore this solution is unique, having identical classical and quantum evolution, as in the case of free particles described by plane waves in ordinary quantum mechanics. The evolution is called "coasting" in the literature and is usually associated with the classic "Milne model" which describes an empty universe. However, the present model is not empty; it is a Friedmann model with $\rho \propto 1/a^2$, capable of describing the expansion of the present universe to a very good approximation.

The approach in this paper is very similar to that in [10], where the evolution of the scale factor in a quantum cosmological model is computed using the dBB formalism. The results are significant, for [10] presents a rigorous proof that the universe could be spontaneously and irreversibly created from nothing,

and that afterwards it proceeds to an inflationary epoch. It is also shown that the exponential expansion stops when the scale factor grows sufficiently large, no matter whether it is closed, open, or flat. It is of particular interest to note that for the open case in this scenario, after the inflationary epoch, the universe enters a coasting evolution $a \propto t$ and continues to expand in this fashion. The spontaneous creation of the coasting universe obtained in the present paper from nothing is an attractive problem worth pursuing. This special case would have benefits such as no singularity, creation of particles by the decay of vacuum energy, a low dark matter/dark energy budget for the present universe, etc.

Solutions having exact classical-quantum correspondence, as obtained in this paper, are important for they save us from the conceptual difficulties in applying the quantum theory to the entire universe.

ACKNOWLEDGMENT

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Annexure VI
UNIVERSITY GRANTS COMMISSION
BAHADUR SHAH ZAFAR MARG, NEW DELHI - 110 002.
Annual/Final Report of the work done on the Minor Research Project.

1. Project report No. 1st /Final: Final
2. UGC Reference No.F.:MRP(S) 1207/11-12/KLMG022/UGC-SWRO
3. Period of report: from 2011 to 2013
4. Title of research project: Modified de Broglie-Bohm Approach to Quantum Cosmology
5. (a) Name of the Principal Investigator: Moncy V. John
   (b) Dept. : Department of Physics
   (c) College where work has progressed: St. Thomas College, Kozhencherry - 689641, Kerala
6. Effective date of starting of the project:

7. Grant approved and expenditure incurred during the period of the report:

   a. Total amount approved Rs.

   b. Total expenditure Rs.

   c. Report of the work done:

   i. Brief objective of the project:

Our attempt in the proposed project was to apply the methods of the modified de Broglie-Bohm approach to quantum mechanics in to a variety of cosmological models and to identify models having the expected classical correspondence. More specifically, the objectives were (a) To formulate the wave equation for the universe (the Wheeler-DeWitt equation) in a form suitable for the modified de Broglie-Bohm approach to quantum mechanics. (b) To solve this equation for various cosmological models, defined in terms of their energy content, equation of state, etc. (c) To find the classical limit of these quantum cosmological formulations in order to identify models with correct classical limit.

   ii. Work done so far and results achieved and publications, if any, resulting from the work (Give details of the papers and names of the journals in which it has been published or accepted for publication:

In this work we found a potential that provides a solution to the WD equation such that the evolution of the universe can be considered classical.

For this, we resorted to trajectory representations and searched for a potential that gives identical classical and quantum evolution for $a(t)$. An equivalent condition for exact quantum-classical correspondence is the vanishing of the quantum potential. We have found that in a Friedmann model where the total energy density varies as $1/a^n$, the exact correspondence happens for $n = 2$. In this case, the WD equation has a simple solution of the form $\Psi = \exp(ima^2/2\hbar)$ and both the dBB and MdBB representations give the result that $a \propto t$. We see that for this kind of total energy/matter density, the classical evolution is also $a \propto t$. Therefore this solution is unique, having identical classical and quantum evolution, as in the case of free particles described by plane waves in ordinary quantum mechanics.

The wave equation for the universe (the Wheeler-DeWitt equation) was formulated in a form suitable for the modified de Broglie-Bohm approach to quantum mechanics. This equation was solved for various cosmological models, defined in terms of their energy content, equation of state, etc. The classical limit of these quantum cosmological formulations were found in order to identify models with correct classical limit.

NO. OF PUBLICATIONS OUT OF THE PROJECT: ONE (M. V. John, *Exact Classical Correspondence in Quantum Cosmology*, Gravitation and Cosmology 2015, Vol. 21, No. 3, pp. 208-215. Pleiades Publishing, Ltd., 2015.)

Paper attached.

iii. Has the progress been according to original plan of work and towards achieving the objective. YES

If not, state reasons: N.A.

iv. Please enclose a summary of the findings of the study. One bound copy of the final report of work done may also be sent to the concerned Regional Office of the UGC.

Writing down the WD equation $\hat{H}\Psi = 0$ in a mini-superspace is one of the simplest applications of quantum theory in problems involving gravity. The dBB and MdBB schemes show that in ordinary quantum mechanics of a single particle, a plane wave solution of the Schrodinger equation corresponds to the motion of a particle that can be considered classical, having a trajectory with constant momentum. This plane wave solution, obtained for a free particle with $V = 0$, is used to describe incoming particles in quantum scattering problems. In this paper we attempt to find a potential that provides a solution to the WD equation such that the evolution of the universe can be considered classical.

For this, we resort to trajectory representations and search for a potential that gives identical classical and quantum evolution for $a(t)$. An equivalent condition for exact quantum-classical correspondence is the vanishing of the quantum potential. We have found that in a Friedmann model where the total energy density varies as $1/a^n$, the exact correspondence happens for $n = 2$. In this case, the WD equation has a simple solution of the form $\Psi = \exp(ima^2/2\hbar)$ and both the dBB and MdBB representations give the result that $a \propto t$. We see that for this kind of total energy/matter density, the classical evolution is also $a \propto t$. Therefore this solution is unique, having identical classical and quantum evolution, as in the case of free particles described by plane waves in ordinary quantum mechanics. The evolution is called 'coasting' in the literature and is usually associated with the classic 'Milne model', which describes an empty universe. However, the present model is not empty; it is a Friedmann model with $\rho \propto 1/a^2$, capable of describing the expansion of the present universe to a very good approximation.

The approach in this work is very similar to that in Gao and Cai (2014), who got the evolution of the scale factor in a quantum cosmological model, computed using the dBB formalism. Their results are significant, for it presents a rigorous proof that the universe could be spontaneously and irreversibly created from nothing, and that after-wards it proceeds to an inflationary epoch. It is also shown that the exponential expansion stops when the scale factor grows sufficiently large, no matter whether it is closed, open, or flat. It is of particular interest to note that for the open case in this scenario, after the inflationary epoch, the universe enters a coasting evolution $a \propto t$ and continues to expand in this fashion. The spontaneous creation of the coasting universe obtained in the present paper from nothing is an attractive problem worth pursuing. This special case would have benefits such as no singularity, creation of particles by the decay of vacuum energy, a low dark matter/dark energy budget for the present universe, etc. Solutions having exact classical-quantum correspondence, as obtained in this work, are important for they save us from the conceptual difficulties in applying the quantum theory to the entire universe.

v. Any other information: Nil

PRINCIPAL INVESTIGATOR   PRINCIPAL
                                              (Seal)